Summary of Special Relativity

MIU PHYS 340 Robert D. Klauber www.quantumfieldtheory.info

Wholeness Chart 1. Overview of the Development of Special Relativity

	Details	**Comment**
Frames considered	Observers in inertial frames only. (But, accelerating object can be handled by special relativity.)	
Chronology		
Pre-Einstein: Galilean transformation	Speed of light: not same for all (not invariant) Laws of nature: **F**=m**a** valid for all observers (invariant) Maxwell's equations not valid for all (not invariant)	
Problems before 1905	Michelson-Morley experiment	Implied light speed not obey Galilean transformation
Einstein postulates	1) Speed of light same for all observers (invariant) 2) Laws of nature same for all observers (invariant in form = covariant)	Invariant in form = vectors in eqs change covariantly
Result of postulate #1	Lorentz transformation (instead of Galilean) $ct' = \frac{1}{\sqrt{1-v^2/c^2}}\left(ct - \frac{v}{c}x\right)$ $\quad ct = \frac{1}{\sqrt{1-v^2/c^2}}\left(ct' + \frac{v}{c}x'\right)$ $x' = \frac{1}{\sqrt{1-v^2/c^2}}(x - vt)$ $\quad x = \frac{1}{\sqrt{1-v^2/c^2}}(x' + vt')$ $y' = y$ $\quad y = y'$ $z' = z$ $\quad z = z'$	Resulted in Lorentz contraction, time dilation, simultaneity not the same for all (not invariant), $E=mc^2$, and more. Reciprocal: each observer sees other frame with Lorentz contraction, time dilation, etc.
Impact on postulate #2	Maxwell's equation valid for all (invariant in form = covariant) **F**=m**a** not valid for all (not invariant in form = not covariant)	
So, Einstein changed mechanics	New 4D law of mechanics: $F^\mu = m\frac{du^\mu}{d\tau}$ where u^μ is 4-velocity	$u^\mu = \frac{dx^\mu}{d\tau}$ τ = proper time on object (see below)
Result of ↑ in 1905	All laws of nature same for all observers (invariant in form = covariant)	Only mechanics and e/m known then.
Result of ↑ up to modern day: Postulate #2 valid	Invariance in form of laws of nature is now a general principle used in all physics. Any law must be covariant under Lorentz transformation.	True for weak and strong force laws. A "must have" for any new proposed theory (SUSY, GUTs, strings, etc.)
Minkowski in 1908	Space and time = 4D spacetime continuum	
	Concepts and Relations	
4D position vector	$x^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$ contravariant components $x_\mu = (-ct, x, y, z) = (x_0, x_1, x_2, x_3) = (-x^0, x^1, x^2, x^3)$ covariant components	
Invariant interval	$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2$ same for all observers between same two events. (We need a minus sign for $(ct)^2$ to get a Lorentz invariant "length" for the position vector between two events.)	Not seen before Minkowski because assumed + sign for $(c\Delta t)^2$. Δs not then invariant.

Proper time τ on an object	Time τ passing on standard clock at rest with respect to object. $\tau = \frac{t}{\gamma} = \sqrt{1-(v/c)^2}\, t \qquad \tau = t'$ at $x' = y' = z' =$ constant	Found from invariant interval between 2 events on object world line.
Proper length L_o of an object	Length measured with meter sticks at rest with respect to object. $L_0 = \gamma L = \frac{1}{\sqrt{1-(v/c)^2}} L \qquad \Delta t' = 0$ and $\Delta t = 0$	Found from invariant interval between 2 events at ends of object at same time in each frame.
4-vector	$w^\mu = (w^0, w^1, w^2, w^3)$ contravariant components $w_\mu = (w_0, w_1, w_2, w_3) = (-w^0, w^1, w^2, w^3)$ covariant components	
Magnitude of a 4-vector	$(w)^2 = \sum_{\mu=0}^{3} w^\mu w_\mu = w^\mu w_\mu = w^0 w_0 + w^1 w_1 + w^2 w_2 + w^3 w_3$ $= -w^0 w^0 + w^1 w^1 + w^2 w^2 + w^3 w^3$	Einstein convention after 2nd equal sign.
Covariance of a 4-vector	To qualify as a legitimate four-vector, its magnitude must be Lorentz invariant. Components can vary, but not magnitude.	
Minkowski metric and 4-vector length	$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (w)^2 = w^\mu w_\mu = \begin{bmatrix} w^0 & w^1 & w^2 & w^3 \end{bmatrix} \begin{bmatrix} -w^0 \\ w^1 \\ w^2 \\ w^3 \end{bmatrix}$ $= \begin{bmatrix} w^0 & w^1 & w^2 & w^3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w^0 \\ w^1 \\ w^2 \\ w^3 \end{bmatrix} = \eta_{\mu\nu} w^\mu w^\nu$ $(s)^2 = \eta_{\mu\nu} x^\mu x^\nu$ (assumes initial $s_0 = 0$, so $\Delta s = s - s_0 = s$)	Compare to $(w)^2$ above Compare to $(\Delta s)^2$ above
4-velocity	$u^\mu = \frac{dx^\mu}{d\tau} = \begin{bmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{bmatrix} = \gamma \begin{bmatrix} c \\ v^1 \\ v^2 \\ v^3 \end{bmatrix} = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \begin{bmatrix} c \\ v^1 \\ v^2 \\ v^3 \end{bmatrix} \qquad v^i = \text{Newtonian velocity} = \frac{dx^i}{dt}$	Always tangent to particle world line
4-velocity squared	$(u)^2 = u^\mu u_\mu = -c^2$ Massive particles.	Invariant. Same for any particle and any observer.
4-momentum	Massive particles $p^\mu = mu^\mu$. ↓ Valid for all particles $p^\mu = \begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{bmatrix} = \begin{bmatrix} E/c \\ p^1 \\ p^2 \\ p^3 \end{bmatrix} \quad E = \frac{mc^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = mc^2 + \underbrace{\tfrac{1}{2}mv^2 + \dots}_{KE_{rel}} \quad p^i = \frac{mv^i}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$	E = relativistic energy; p^i = relativistic 3-momentum
4-momentum squared	$(p)^2 = p^\mu p_\mu = -m^2c^2$ Massive and massless particles. $p^\mu p_\mu = -\frac{E^2}{c^2} + p^i p_i = -\frac{E'^2}{c^2} + p'^i p'_i = p'^\mu p'_\mu = -m^2c^2$	Invariant. Same for any observer, any velocity. Different for different mass.
4D unit basis vectors	$\boldsymbol{e}_\mu = \boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2$, and $\boldsymbol{e}_3$.	Like **i**, **j**, **k** in 3D
4-vectors	$\mathbf{A} = A^0\boldsymbol{e}_0 + A^1\boldsymbol{e}_1 + A^2\boldsymbol{e}_2 + A^3\boldsymbol{e}_3 = A^\mu \boldsymbol{e}_\mu$ Same **A**, diff frames: $A^\mu \boldsymbol{e}_\mu = A'^\mu \boldsymbol{e}'_\mu$	
Invariance vs conservation	Invariance = no change for different coordinate systems (observers) Conservation = no change over time	Δs invariant, not conserved E conserved, not invariant

Spacetime Diagrams	See figures below.		
Concept/Entity	Timelike Interval (AB, Fig. 1)	Spacelike Interval (AC)	Lightlike Interval (AD)
Region on spacetime diagram	Inside light cone	Outside light cone	On surface of light cone
Space vs time components	$c\Delta t > \Delta x$	$c\Delta t < \Delta x$	$c\Delta t = \Delta x$
$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2$	negative	positive	zero (null)
Δs	imaginary	real	zero
Travel from first event to second?	Yes	No	Only light can.
Find proper time τ from $(\Delta s)^2 = (c\tau)^2$?	Yes, for a particle traveling from first to second event	No. Particle would have to travel faster than light.	$\tau = 0$ for a photon.
Can first event affect (cause) the second event?	Yes. At or below light speed, a signal from 1st event can reach 2nd event.	No. A signal would have to travel faster than light.	Yes, but only an electromagnetic signal. All others too slow.
Can the two events be simultaneous for some observer?	No. Inside light cone can never have the x' axis (the axis where all events occur at the same time)	Yes. Can have x' axis extending from first event to second.	No. Observer would have to travel at light speed between events to see them simultaneous.
Is the time order of events the same for all observers?	Yes	No. (See Fig. 2 below.)	Yes
Δs invariant?	Yes	Yes	Yes
Is the primed time axis timelike, spacelike, or null?	Yes, timelike.	No.	No.
Is the primed space axis timelike, spacelike, or null?	No.	Yes, spacelike.	No.
Is the light path for all observers timelike, spacelike, or null?	No.	No.	Yes, null.

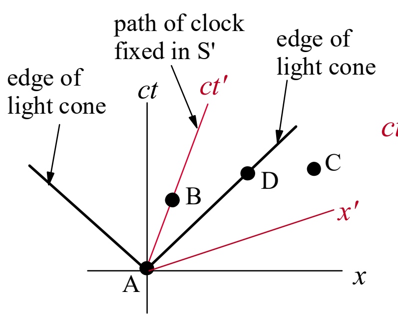

Figure 11-1 Kinds of Intervals

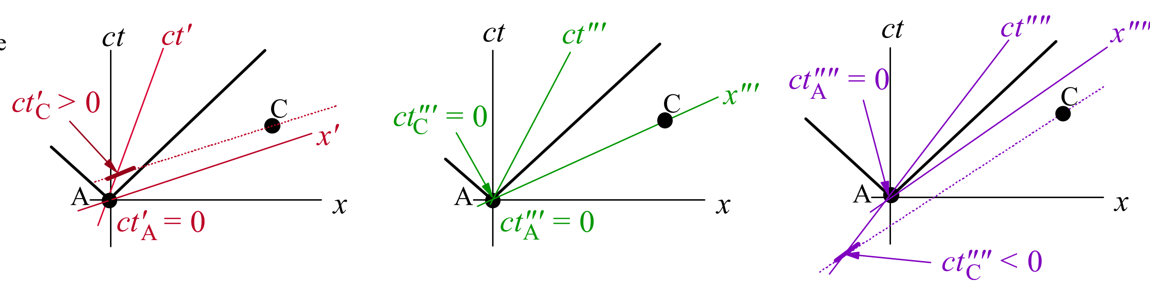

Figure 11-2. Order of Spacelike Separated Events Different for Different Observers
(Not for timelike such as AB. Cannot rotate space axis through both events, so never simultaneous nor reversed in order.)

Wholeness Chart 2. Electromagnetism: Classical 3D + 1 vs Relativistic 4D Spacetime

Equation numbers are for Griffiths, *Introduction to Electromagnetism*, 4th ed.

Entity	**3D + 1**	**4D**	**Comment**
E/m field tensor	N/A	$F^{\mu\nu} = \begin{bmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & B^3 & -B^2 \\ -E^2/c & -B^3 & 0 & B^1 \\ -E^3/c & B^2 & -B^1 & 0 \end{bmatrix}$ (12.119)	Antisymmetric. $F^{\mu\nu} = F^{\mu\nu}(t,x^i) = F^{\mu\nu}(x^\beta)$
E/m dual field tensor	N/A	$G^{\mu\nu} = \begin{bmatrix} 0 & B^1 & B^2 & B^3 \\ -B^1 & 0 & -E^3/c & E^2/c^2 \\ -B^2 & E^3/c^2 & 0 & -E^1/c \\ -B^3 & -E^2/c^2 & E^1/c & 0 \end{bmatrix}$ (12.120)	Antisymmetric, $G^{\mu\nu} = G^{\mu\nu}(t,x^i) = G^{\mu\nu}(x^\beta)$
Electric field	3-vector **E** or $\boldsymbol{E^i}$	cF^{01}, cF^{02}, cF^{03}	Components of the 4D e/m field tensor
Magnetic field	3-vector **B** or $\boldsymbol{B^i}$	F^{23}, F^{31}, F^{12}	
Charge	Q	Q	Invariant
Proper charge density	$\rho = \rho_0 = \dfrac{Q}{V_0} = \dfrac{Q}{V}$ $(V \rightarrow 0)$ page 565	$\rho_0 = \dfrac{Q}{V_0}$ $(V \rightarrow 0)$ (12.121), (12.122)	Invariant, $\rho = \rho(t,x^i) = \rho(x^\mu)$
Charge density	$\rho = \dfrac{Q}{V}$ same as above	$\rho = \dfrac{Q}{V} = \dfrac{Q}{V_0\sqrt{1-v^2/c^2}} = \dfrac{\rho_0}{\sqrt{1-v^2/c^2}}$ (12.122)	Not 4D invariant, since V not
Current density	$\mathbf{J} = \rho\,\mathbf{v}$ page 565	$\mathbf{J} = \rho\mathbf{v} = \dfrac{\rho_0\mathbf{v}}{\sqrt{1-v^2/c^2}} = \rho_0 u^i$ (12.122)	Griffiths uses **u** for our **v**, η for our u
4-current	N/A	$J^\mu = [J^0, J^1, J^2, J^3] = [\rho c, \rho v^1, \rho v^2, \rho v^3]$ (12.124) $= \rho_0 u^\mu$ $J_\mu = [-\rho c, \rho v^1, \rho v^2, \rho v^3] = \rho_0 u_\mu$	Not 4D invariant, $J^\mu = J^\mu(t,x^i) = J^\mu(x^\mu)$
Continuity equation (charge)	$\nabla \bullet \mathbf{J} + \dfrac{\partial \rho}{\partial t} = 0$ below (12.124)	$\dfrac{\partial J^\mu}{\partial x^\mu} = 0$ (12.126)	$x^0 = ct$
Maxwell's equations	$\nabla \cdot \mathbf{E} = \dfrac{1}{\varepsilon_0}\rho$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ (7.40) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$	$\dfrac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu$ $\dfrac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$ (12.127)	
Lorentz force law	$\mathbf{F} = q\left(\mathbf{E} + (\mathbf{v} \times \mathbf{B})\right)$ below (12.129)	$K^\mu = q u_\nu F^{\mu\nu}$ (12.128)	Griffith's K^μ is our F^μ
Scalar & vector potential	Φ and **A**	$A^\mu = [\Phi/c, A^i]^{\mathrm{T}} = [A^0, A^1, A^2, A^3]$ (12.132)	Griffith's V is our Φ $A^\mu = A^\mu(x^\nu)$

E/m field tensor via 4-potential	N/A	$F^{\mu\nu} = \frac{\partial A^{\mu}}{\partial x_{\nu}} - \frac{\partial A^{\nu}}{\partial x_{\mu}}$ (12.133)	Antisymmetric. $F^{\mu\nu} = F^{\mu\nu}(x^{\beta})$
Maxwell's equations via 4-potential	$\frac{\partial}{c\partial t}(\nabla\cdot\mathbf{A}) - \nabla^2\Phi = c\mu_0\rho$ $-\nabla\left(\frac{\partial\Phi}{c\partial t} + (\nabla\cdot\mathbf{A})\right) + \frac{\partial^2\mathbf{A}}{c^2\partial t^2} - \nabla^2\mathbf{A} = \mu_0\mathbf{J}$	$\frac{\partial}{\partial x_{\mu}}\left(\frac{\partial A^{\nu}}{\partial x^{\nu}}\right) - \frac{\partial}{\partial x_{\nu}}\left(\frac{\partial A^{\mu}}{\partial x^{\nu}}\right) = \mu_0 J^{\mu}$ (12.134)	For 3D + 1, different units, see Jackson 1975 pg 220 (6.32), (6.33)
Lorenz gauge	$\nabla\bullet\mathbf{A} = -\frac{1}{c}\frac{\partial\Phi}{\partial t}$ pg. 570	$\frac{\partial A^{\nu}}{\partial x^{\nu}} = 0$ (12.136)	$\Phi = V/c$
Maxwell's equations in Lorenz gauge	$-\frac{\partial^2\Phi}{c^2\partial t^2} - \nabla^2\Phi = c\mu_0\rho$ $\frac{\partial^2\mathbf{A}}{c^2\partial t^2} - \nabla^2\mathbf{A} = \mu_0\mathbf{J}$	$-\frac{\partial}{\partial x_{\nu}}\frac{\partial}{\partial x^{\nu}}A^{\mu} = \mu_0 J^{\mu}$ (12.137)	For 3D + 1, different units, see Jackson 1975 pg 220 (6.37), (6.38)

NOTE: For no source charges or 3-currents (i.e., $J^{\mu} = 0$) last equation above ((12.137) in Griffiths) is just the wave equation

$$\frac{\partial}{\partial x_{\nu}}\frac{\partial}{\partial x^{\nu}}A^{\mu} = 0 \xrightarrow[\text{in } x^1 \text{ direction}]{\text{for } \mu=2} \frac{\partial}{\partial x_0}\frac{\partial}{\partial x^0}A^2 + \frac{\partial}{\partial x_1}\frac{\partial}{\partial x^1}A^2 \underbrace{+\frac{\partial}{\partial x_2}\frac{\partial}{\partial x^2}A^2 + \frac{\partial}{\partial x_3}\frac{\partial}{\partial x^3}A^2}_{=0 \text{ since } A^2 \text{ only depends on } x^1, t} = 0$$

$$\xrightarrow{\text{take } x^1 \text{ as } x} -\frac{\partial^2 A^2}{c^2\partial t^2} + \frac{\partial^2 A^2}{\partial x^2} = 0 \quad \rightarrow \quad \frac{\partial^2 A^2}{\partial t^2} = c^2\frac{\partial^2 A^2}{\partial x^2} \qquad c = \text{wave speed}$$